Math 10B with Professor Stankova
Worksheet, Discussion #25; Thursday, 4/25/2019
GSI name: Roy Zhao

T-Test

Concepts

1. You use a T test if your sample size is small $n < 30$. The **t-statistic** for a value x is $\frac{|x-\mu|}{\sigma/\sqrt{n}}$, the same value we calculated for the z test. The **degrees of freedom** is $\nu = n-1$. The PDF of the t distribution with ν degrees of freedom is

$$f_\nu(x) = \frac{1}{\sqrt{\nu\pi}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

The **Γ-function** is $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. It satisfies the property that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, $\Gamma(n) = (n-1)!$ for $n \geq 1$, and $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

Examples

2. An infomercial claims that a miracle drug will cause you to grow all your hair back. There are 25 brave participants and surprisingly 7 people regrew their hair. If normally 10% of people regrow their hair, can you say that this drug worked?

Solution: We would expect that 10% of people will regrow their hair with standard deviation $\sigma = \sqrt{p(1-p)} = \sqrt{0.1(0.9)} = 0.3$. There are $7/25 = 28\%$ who regrew their hair. The t statistic is $t(|0.28-0.1|/(0.3/\sqrt{25})) = t(3)$. Then in this case we have $25-1 = 24$ degrees of freedom and looking at the t score table with $\nu = 24$ gives $t(3) = 0.003 < \alpha$. Therefore, we can reject the null hypothesis and say that this drug does help you grow your hair back.

Problems

3. True **FALSE** As the number of degrees of freedom increase the probability that $P(T \geq t) \to 0$ for fixed t.

> **Solution:** As the degrees of freedom increase, then the T distribution becomes closer to a normal distribution so $P(T \geq t) \to P(Z \geq t)$.

4. True **FALSE** For large degrees of freedom, the values gotten from the t table are the same as those from a z score table.

> **Solution:** The values get closer and closer to the z score table but technically will never be equal.

5. True **FALSE** The PDF of the t distribution is only defined for positive values of x.

> **Solution:** It is defined for all x but it is symmetric across 0.

6. True **FALSE** It is possible to do a t test with only one measurement.

> **Solution:** If there is one measurement then $\nu = 0$ and this is undefined.

7. You think that legacy students are being admitted to college with lower test scores than non-legacy students. Suppose that the average SAT score of non-legacy students is $\mu = 1300$. You ask 9 legacy students what their SAT scores were and get a sample mean of $\bar{x} = 1270$ and a sample standard deviation of $s = 90$. Can you say that legacy students are being admitted with lower standards with $\alpha = 10\%$?

> **Solution:** We want to calculate the probability that $P(X \leq 1280)$ and the t statistic is $t(\frac{|1270-1300|}{90/\sqrt{9}}) = t(\frac{30}{30}) = t(1)$. We have 8 degrees of freedom and looking this up gives $t(1) = 0.173 > \alpha$. So we cannot reject the null hypothesis that the average for legacy students should be the same.

8. You think that legacy students are different from non-legacy students. Suppose that the average SAT score of non-legacy students is $\mu = 1300$. You ask 9 legacy students what their SAT scores were and get a sample mean of $\bar{x} = 1350$ and a sample standard deviation of $s = 75$. Can you say that legacy students are different from non-legacy students with $\alpha = 0.05$?

Solution: We want to calculate the probability that $P(X \geq 1350)$ and the t statistic is $t(\frac{|1350-1300|}{75/\sqrt{9}}) = t(\frac{50}{25}) = t(2)$. We have 8 degrees of freedom and looking this up gives $t(2) = 0.040 > \alpha/2 = 0.025$. So we cannot reject the null hypothesis that the average for legacy students should be the same.

9. The heart rates of 4 patients in an ICU have mean 96 beats per minute and standard deviation 16. Are the heart rates for ICU patients unusual given that the normal heart rate has a mean of 72 beats per minutes with $\alpha = 0.05$?

Solution: We want to calculate the probability that $P(X \leq 88)$ and the t statistic is $t(\frac{|96-72|}{16/\sqrt{4}}) = t(\frac{24}{8}) = t(3)$. We have 3 degrees of freedom and looking this up gives $t(3) = 0.029 < \alpha$. So we reject the null hypothesis that the heart rate is no different.

10. Calculate $\Gamma(4)$ and $\Gamma(5/2)$.

Solution:

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2\Gamma(2) = 3 \cdot 2 \cdot 1\Gamma(1) = 6 \cdot 1 = 6.$$

$$\Gamma(5/2) = \frac{5}{2}\Gamma(3/2) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(1/2) = \frac{15}{8}\sqrt{\pi}.$$

11. Write the PDF for the t distribution when $\nu = 4$.

Solution:

$$f_4(x) = \frac{1}{\sqrt{\nu\pi}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \frac{1}{\sqrt{4\pi}}\frac{\Gamma(5/2)}{\Gamma(2)}(1 + x^2/4)^{-5/2}$$

$$= \frac{15\sqrt{\pi}/8}{1 \cdot 2\sqrt{\pi}}(1 + x^2/4)^{-5/2} = \frac{15}{16}(1 + x^2/4)^{-5/2}.$$

12. Prove that $\Gamma(x + 1) = x\Gamma(x)$ for all $x > 0$.

Solution: We integrate by parts with $u = t^x$ and $dv = e^{-t}dt$ so $du = xt^{x-1}$ and $v = -e^{-t}$ to get

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt \\ &= t^x(-e^{-t})|_0^\infty + \int_0^\infty xt^{x-1}e^{-t}dt \\ &= 0 - 0 + x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x).\end{aligned}$$

13. Use induction to prove that $\Gamma(n) = (n-1)!$ for all $n \geq 1$.

Solution: We prove the base case of $n = 1$. We get $\Gamma(1) = \int_0^\infty t^0 e^{-t}dt = \int_0^\infty e^{-t}dt = 1 = 0!$. Now assume the inductive hypothesis for some $n \geq 1$. Then, we have that

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n! = (n+1-1)!.$$

Thus by mathematical induction, we have shown the result for all $n \geq 1$.

Goodness of Fit Testing

14. You use a χ^2 test to determine if a distribution is how you expect it to be. Suppose that you expect it to be distributed with a different values and for each of these values, you expect to get outcome k m_k times but actually get it n_k times. Then you compare the statistic

$$r = \sum_{k=1}^{a} \frac{(n_k - m_k)^2}{m_k} = \sum \frac{(\text{observed - expected})^2}{\text{expected}}$$

with the $\chi^2_{k=a-1}$ distribution. The χ^2 distribution with $k = a-1$ degrees of freedom has the PDF for $x \geq 0$

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1}e^{-x/2}.$$

Example

15. In a skittle bag, you get 11 red skittles, 12 blue, 5 green, 10 yellow, and 12 orange skittles. Is it possible that the colors are evenly distributed with a significance level of $\alpha = 0.05$?

Solution: In 50 skittles, we expect to get 10 of each. Following the formula, our statistic is:

$$\frac{(11-10)^2}{10}+\frac{(12-10)^2}{10}+\frac{(5-10)^2}{10}+\frac{(10-10)^2}{10}+\frac{(12-10)^2}{10}$$
$$=\frac{1+4+25+0+4}{10}=3.4.$$

There are 5 options so we have $5-1=4$ degrees of freedom. For 4 degrees of freedom and $\alpha=0.05$, our critical value is 9.488. Since $3.4<9.488$, we cannot reject the null hypothesis that the colors are evenly distributed.

Problems

16. **TRUE** False If the critical value for the χ^2 distribution with k degrees of freedom is r, then $P(R\geq r)=\alpha$.

Solution: This is the definition of the critical value.

17. **TRUE** False For fixed significance level α, as the number of degrees of freedom increases, the critical value also increases.

18. You take 400 cards and get 100 spades, 105 hearts, 107 diamonds, and 88 clubs. Can you say that the suits are not evenly distributed with $\alpha=0.05$?

Solution: If they are evenly distributed, we would expect 100 of each. Calculating the statistic gives

$$r=\frac{(100-100)^2}{100}+\frac{(105-100)^2}{100}+\frac{(107-100)^2}{100}+\frac{(88-100)^2}{100}$$
$$=\frac{25+49+144}{100}=2.18.$$

For $4-1=3$ degrees of freedom, our critical value for α is 7.815 and since $2.18<7.815$, we cannot reject the null hypothesis.

19. You expect to get a distribution of brown eyes brown hair to brown eyes blond hair to blue eyes brown hair to blue eyes blond hair as $9:3:3:1$. When looking around in class, you get a distribution of $61:19:11:9$ after looking at 100 people. Is this distribution accurate (use $\alpha=0.05$)?

Solution: If it were accurately distributed, we would expect 56.25 : 18.75 : 18.75 : 6.25. So computing the statistic gives

$$r = \frac{(61-56.25)^2}{56.25} + \frac{(19-18.75)^2}{18.75} + \frac{(11-18.75)^2}{18.75} + \frac{(9-6.25)^2}{9.25}$$
$$\approx 4.43.$$

Since $4.43 < 7.815$ which is the critical value for $4-1=3$ degrees of freedom and $\alpha = 0.05$, we cannot reject the null hypothesis.

20. Find $\chi^2(x)$ for $k = 1, 2, 4$.

Solution:

$$\chi^2_{k=1}(x) = \frac{1}{2^{1/2}\Gamma(1/2)} x^{1/2-1} e^{-x/2} = \frac{1}{\sqrt{2}\sqrt{\pi}} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{e^x}} = \frac{1}{\sqrt{2\pi x e^x}}.$$

$$\chi^2_{k=2}(x) = \frac{1}{2^{2/2}\Gamma(2/2)} x^{2/2-1} e^{-x/2} = \frac{1}{2 \cdot 0!} x^0 e^{-x/2} = \frac{1}{2} e^{-x/2}.$$

$$\chi^2_{k=4}(x) = \frac{1}{2^{4/2}\Gamma(4/2)} x^{4/2-1} e^{-x/2} = \frac{1}{4 \cdot 1!} x^1 e^{-x/2} = \frac{1}{4} x e^{-x/2}.$$

21. Use induction to prove that $E[\chi^2_{k=2n}(x)] = 2n$ for all $n \geq 1$.

Solution: Show the base case then use integration by parts with $u = x^n$. Come to office hours for details.

p-values for one-tailed t-distribution $= \int_{\mathbf{t}}^{\infty} \mathbf{f}_{\nu}(\mathbf{x})\mathbf{dx}$

$\nu = n - 1$ degrees of freedom $t = t$-score

$t \downarrow$ $\nu \rightarrow$	1	2	3	4	5	6	7	8	9	10
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.2	0.437	0.430	0.427	0.426	0.425	0.424	0.424	0.423	0.423	0.423
0.4	0.379	0.364	0.358	0.355	0.353	0.352	0.351	0.350	0.349	0.349
0.6	0.328	0.305	0.295	0.290	0.287	0.285	0.284	0.283	0.282	0.281
0.8	0.285	0.254	0.241	0.234	0.230	0.227	0.225	0.223	0.222	0.221
1.0	0.250	0.211	0.196	0.187	0.182	0.178	0.175	0.173	0.172	0.170
1.2	0.221	0.177	0.158	0.148	0.142	0.138	0.135	0.132	0.130	0.129
1.4	0.197	0.148	0.128	0.117	0.110	0.106	0.102	0.100	0.098	0.096
1.6	0.178	0.125	0.104	0.092	0.085	0.080	0.077	0.074	0.072	0.070
1.8	0.161	0.107	0.085	0.073	0.066	0.061	0.057	0.055	0.053	0.051
2.0	0.148	0.092	0.070	0.058	0.051	0.046	0.043	0.040	0.038	0.037
2.2	0.136	0.079	0.058	0.046	0.040	0.035	0.032	0.029	0.028	0.026
2.4	0.126	0.069	0.048	0.037	0.031	0.027	0.024	0.022	0.020	0.019
2.6	0.117	0.061	0.040	0.030	0.024	0.020	0.018	0.016	0.014	0.013
2.8	0.109	0.054	0.034	0.024	0.019	0.016	0.013	0.012	0.010	0.009
3.0	0.102	0.048	0.029	0.020	0.015	0.012	0.010	0.009	0.007	0.007
3.2	0.096	0.043	0.025	0.016	0.012	0.009	0.008	0.006	0.005	0.005
3.4	0.091	0.038	0.021	0.014	0.010	0.007	0.006	0.005	0.004	0.003
3.6	0.086	0.035	0.018	0.011	0.008	0.006	0.004	0.003	0.003	0.002
3.8	0.082	0.031	0.016	0.010	0.006	0.004	0.003	0.003	0.002	0.002
4.0	0.078	0.029	0.014	0.008	0.005	0.004	0.003	0.002	0.002	0.001
4.2	0.074	0.026	0.012	0.007	0.004	0.003	0.002	0.001	0.001	0.001
4.4	0.071	0.024	0.011	0.006	0.004	0.002	0.002	0.001	0.001	0.001
4.6	0.068	0.022	0.010	0.005	0.003	0.002	0.001	0.001	0.001	0.000
4.8	0.065	0.020	0.009	0.004	0.002	0.002	0.001	0.001	0.000	0.000
5.0	0.063	0.019	0.008	0.004	0.002	0.001	0.001	0.001	0.000	0.000
5.2	0.060	0.018	0.007	0.003	0.002	0.001	0.001	0.000	0.000	0.000
5.4	0.058	0.016	0.006	0.003	0.001	0.001	0.001	0.000	0.000	0.000
5.6	0.056	0.015	0.006	0.002	0.001	0.001	0.000	0.000	0.000	0.000
5.8	0.054	0.014	0.005	0.002	0.001	0.001	0.000	0.000	0.000	0.000
6.0	0.053	0.013	0.005	0.002	0.001	0.000	0.000	0.000	0.000	0.000
6.2	0.051	0.013	0.004	0.002	0.001	0.000	0.000	0.000	0.000	0.000
6.4	0.049	0.012	0.004	0.002	0.001	0.000	0.000	0.000	0.000	0.000
6.6	0.048	0.011	0.004	0.001	0.001	0.000	0.000	0.000	0.000	0.000
6.8	0.046	0.010	0.003	0.001	0.001	0.000	0.000	0.000	0.000	0.000
7.0	0.045	0.010	0.003	0.001	0.000	0.000	0.000	0.000	0.000	0.000
7.2	0.044	0.009	0.003	0.001	0.000	0.000	0.000	0.000	0.000	0.000
7.4	0.043	0.009	0.003	0.001	0.000	0.000	0.000	0.000	0.000	0.000
7.6	0.042	0.008	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
7.8	0.041	0.008	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
8.0	0.040	0.008	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
8.2	0.039	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
8.4	0.038	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
8.6	0.037	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000
8.8	0.036	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.0	0.035	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.2	0.034	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.4	0.034	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.6	0.033	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.8	0.032	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.032	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
32	15.13	16.36	18.29	20.07	22.27	42.58	46.19	49.48	53.49
34	16.50	17.79	19.81	21.66	23.95	44.90	48.60	51.97	56.06
38	19.29	20.69	22.88	24.88	27.34	49.51	53.38	56.90	61.16
42	22.14	23.65	26.00	28.14	30.77	54.09	58.12	61.78	66.21
46	25.04	26.66	29.16	31.44	34.22	58.64	62.83	66.62	71.20
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
55	31.73	33.57	36.40	38.96	42.06	68.80	73.31	77.38	82.29
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
65	39.38	41.44	44.60	47.45	50.88	79.97	84.82	89.18	94.42
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43
75	47.21	49.48	52.94	56.05	59.79	91.06	96.22	100.84	106.39
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33
85	55.17	57.63	61.39	64.75	68.78	102.08	107.52	112.39	118.24
90	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12
95	63.25	65.90	69.92	73.52	77.82	113.04	118.75	123.86	129.97
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81